

SIXTH TERM EXAMINATION PAPER
administered by the Oxford and Cambridge Schools Examination Board
on behalf of the Cambridge Colleges

9465

MATHEMATICS

Friday 26 June 1992, afternoon

3 hours

Additional materials:

script paper; graph paper; MF(STEP)1.

*To be brought by candidate: electronic calculator;
standard geometrical instruments.*

All questions carry equal weight.

You are reminded that extra credit is given for complete answers and that little credit is given for isolated fragments.

You may attempt as many questions as you wish with no restriction of choice but marks will be assessed on the six questions best answered.

You are provided with Mathematical Formulae and Tables MF(STEP)1.

The use of Electronic calculators is permitted.

Section A: Pure Mathematics

1 Today's date is June 26th 1992 and the day of the week is Friday. Find which day of the week was April 3rd 1905, explaining your method **carefully**.

[30 days hath September, April, June and November. All the rest have 31, excepting February alone which has 28 days clear and 29 in each leap year.]

2 A 3×3 magic square is a 3×3 array

$$\begin{array}{ccc} a & b & c \\ d & e & f \\ g & h & k \end{array}$$

whose entries are the nine distinct integers 1, 2, 3, 4, 5, 6, 7, 8, 9 and which has the property that all its rows, columns and main diagonals add up to the same number n . (Thus $a + b + c = d + e + f = g + h + k = a + d + g = b + e + h = c + f + k = a + e + k = c + e + g = n$.)

(i) Show that $n = 15$.

(ii) Show that $e = 5$.

(iii) Show that one of b, d, h or f must have value 9.

(iv) Find all 3×3 magic squares with $b = 9$.

(v) How many different 3×3 magic squares are there? Why?

[Two magic squares are different if they have different entries in any place of the array.]

3 Evaluate

(i) $\int_{-\pi}^{\pi} |\sin x| dx,$

(ii) $\int_{-\pi}^{\pi} \sin |x| dx,$

(iii) $\int_{-\pi}^{\pi} x \sin x dx,$

(iv) $\int_{-\pi}^{\pi} x^{10} \sin x dx.$

4 Sketch the following subsets of the complex plane using Argand diagrams. Give reasons for your answers.

(i) $\{z : \operatorname{Re}((1+i)z) \geq 0\}.$

(ii) $\{z : |z^2| \leq 2, \operatorname{Re}(z^2) \geq 0\}.$

(iii) $\{z = z_1 + z_2 : |z_1| = 2, |z_2| = 1\}.$

5 Let $p_0(x) = (1-x)(1-x^2)(1-x^4)$. Show that $(1-x)^3$ is a factor of $p_0(x)$. If $p_1(x) = xp'_0(x)$ show, by considering factors of the polynomials involved, that $p'_0(1) = 0$ and $p'_1(1) = 0$.

By writing $p_0(x)$ in the form

$$p_0(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + c_6x^6 + c_7x^7,$$

deduce that

$$\begin{aligned} 1 + 2 + 4 + 7 &= 3 + 5 + 6, \\ 1^2 + 2^2 + 4^2 + 7^2 &= 3^2 + 5^2 + 6^2. \end{aligned}$$

Show that we can write the integers $1, 2, \dots, 15$ in some order as a_1, a_2, \dots, a_{15} in such a way that

$$a_1^r + a_2^r + \dots + a_8^r = a_9^r + a_{10}^r + \dots + a_{15}^r$$

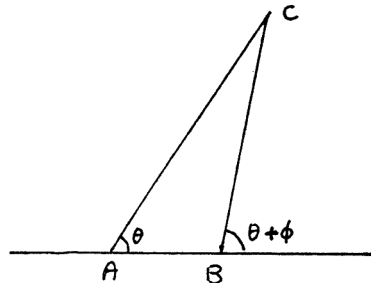
for $r = 1, 2, 3$.

6 Explain briefly, by means of a diagram, or otherwise, why

$$f(\theta + \delta\theta) \approx f(\theta) + f'(\theta)\delta\theta,$$

when $\delta\theta$ is small.

Two powerful telescopes are placed at points A and B which are a distance a apart. A very distant point C is such that AC makes an angle θ with AB and BC makes an angle $\theta + \phi$ with AB produced. (A sketch of the arrangement is given in the diagram.)



If the perpendicular distance h of C from AB is very large compared with a show that h is approximately $(a \sin^2 \theta)/\phi$ and find the approximate value of AC in terms of a , θ and ϕ .

It is easy to show (but you are not asked to show it) that errors in measuring ϕ are much more important than errors in measuring θ . If we make an error of $\delta\phi$ in measuring ϕ (but measure θ correctly) what is the approximate error in our estimate of AC and, roughly, in what proportion is it reduced by doubling the distance between A and B ?

7 Let $g(x) = ax + b$. Show that, if $g(0)$ and $g(1)$ are integers, then $g(n)$ is an integer for all integers n .

Let $f(x) = Ax^2 + Bx + C$. Show that, if $f(-1)$, $f(0)$ and $f(1)$ are integers, then $f(n)$ is an integer for all integers n .

Show also that, if α is any real number and $f(\alpha - 1)$, $f(\alpha)$ and $f(\alpha + 1)$ are integers, then $f(\alpha + n)$ is an integer for all integers n .

8 Explain diagrammatically, or otherwise, why

$$\frac{d}{dx} \int_a^x f(t) dt = f(x).$$

Show that, if

$$f(x) = \int_0^x f(t) dt + 1,$$

then $f(x) = e^x$.

What is the solution of

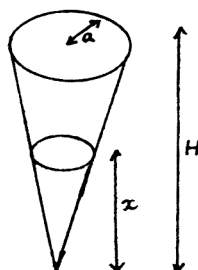
$$f(x) = \int_0^x f(t) dt ?$$

Given that

$$\int_0^x f(t) dt = \int_x^1 t^2 f(t) dt + x - \frac{x^5}{5} + C,$$

find $f(x)$ and show that $C = -2/15$.

9 The diagram shows a coffee filter consisting of an inverted hollow right circular cone of height H cm and base radius a cm.



When the water level is x cm above the vertex, water leaves the cone at a rate Ax $\text{cm}^3\text{sec}^{-1}$, where A is a positive constant. Suppose that the cone is initially filled to a height h cm with $0 < h < H$. Show that it will take $\pi a^2 h^2 / (2AH^2)$ seconds to empty.

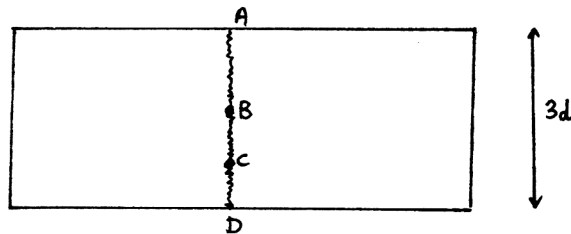
Suppose now that the cone is initially filled to a height h cm, but that water is poured in at a constant rate B $\text{cm}^3 \text{sec}^{-1}$ and continues to drain as before. Establish, by considering the sign of $\frac{dx}{dt}$, or otherwise, what will happen subsequently to the water level in the different cases that arise. (You are not asked to find an explicit formula for x .)

Section B: Mechanics

10 A projectile of mass m is fired horizontally from a toy cannon of mass M which slides freely on a horizontal floor. The length of the barrel is l and the force exerted on the projectile has the constant value P for so long as the projectile remains in the barrel. Initially the cannon is at rest. Show that the projectile emerges from the barrel at a speed relative to the ground of

$$\left(\frac{2PMl}{m(M+m)} \right)^{1/2}.$$

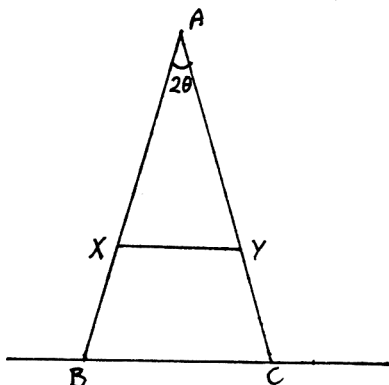
11 Three light elastic strings AB , BC and CD , each of natural length a and modulus of elasticity λ , are joined together as shown in the diagram.



A is attached to the ceiling and D to the floor of a room of height $3d$ in such a way that A , B , C and D are in a vertical line. Particles of mass m are attached at B and C . Find the heights of B and C above the floor.

Find the set of values of d for which it is possible, by choosing m suitably, to have $CD = a$?

12 The diagram shows a crude step-ladder constructed by smoothly hinging together two light ladders AB and AC , each of length l , at A . A uniform rod of wood, of mass m , is pin-jointed to X on AB and to Y on AC , where $AX = 3l/4 = AY$. The angle $\angle XAY$ is 2θ .



The rod XY will break if the tension in it exceeds T . The step-ladder stands on rough horizontal ground (coefficient of friction μ). Given that $\tan \theta > \mu$, find how large a mass M can safely be placed at A and show that if

$$\tan \theta > \frac{6T}{mg} + 4\mu$$

the step-ladder will fail under its own weight.

[You may assume that friction is limiting at the moment of collapse.]

13 A comet, which may be regarded as a particle of mass m , moving in the sun's gravitational field, at a distance x from the sun, experiences a force Gmx^{-2} (where G is a constant) directly towards the sun. Show that if, at some time, $x = h$ and the comet is travelling directly away from the sun with speed V , then x cannot become arbitrarily large unless $V^2 \geq 2Gh^{-1}$.

A comet is initially motionless at a great distance from the sun. If, at some later time, it is at a distance h from the sun, how long after that will it take to fall into the sun?

Section C: Probability and Statistics

14 The average number of pedestrians killed annually in road accidents in Poldavia during the period 1974–1989 was 1080 and the average number killed annually in commercial flight accidents during the same period was 180. Discuss the following newspaper headlines which appeared in 1991. (The percentage figures in square brackets give a rough indication of the weight of marks attached to each discussion.)

- (i) [10%] Six Times Safer To Fly Than To Walk. 1974–1989 Figures Prove It.
- (ii) [10%] Our Skies Are Safer. Only 125 People Killed In Air Accidents In 1990.
- (iii) [30%] Road Carnage Increasing. 7 People Killed On Tuesday.
- (iv) [50%] Alarming Rise In Pedestrian Casualties. 1350 Pedestrians Killed In Road Accidents During 1990.

15 Trains leave Barchester Station for London at 12 minutes past the hour, taking 60 minutes to complete the journey and at 48 minutes past the hour, taking 75 minutes to complete the journey. The arrival times of passengers for London at Barchester Station are uniformly distributed over the day and all passengers take the first available train. Show that their average journey time from arrival at Barchester Station to arrival in London is 84.6 minutes.

Suppose that British Rail decide to retime the fast 60 minute train so that it leaves at x minutes past the hour. What choice of x will minimise the average journey time?

16 The four towns A , B , C and D are linked by roads AB , AC , CB , BD and CD . The probability that any one road will be blocked by snow on the 1st of January is p , independent of what happens to any other [$0 < p < 1$]. Show that the probability that the only open route from A to D is $ABCD$ is $p^2(1 - p)^3$. In order to increase the probability that it is possible to get from A to D by a sequence of unblocked roads the government proposes either to snow-proof the road AB (so that it can never be blocked) or to snow-proof the road CB . Because of the high cost it cannot do both. Which road should it choose (or are both choices equally advantageous)?

In fact, $p = 1/10$ and the government decides that it is only worth going ahead if the present probability of A being cut off from D is greater than $1/100$. Will it go ahead?